**Recitation 8**

**Problem 1: Build-up error**

**False claim:** If every vertex in a graph has positive degree, then the graph is connected.

1. Prove that this claim is indeed false by providing a counterexample.

**Proof** by counterexample. Consider a graph with four nodes, . If then every node has a degree of one, a positive degree, and yet there is no path between , thus it is not connected.

2. Find the mistake in the proof listed on the assignment.

“Since x has positive degree, there is an edge from x to some other vertex, y.”

This is the first logical mistake. The positive degree property of the graph cannot be assumed after adding x.

I was almost correct with this, but the problem was sort of a trick. The part I selected is not logically incorrect, but rather too restrictive. It narrows the proof to the topic of only showing how connected graphs remain connected when a node of positive degree is added. But what needs to be shown is that **all -vertex positive-degree graphs** must be connected, which includes graphs that can’t be produced by adding more positive-degree vertices. The graph in my counterexample, for example, cannot be built up from a positive-degree three-node graph. Therefore, the logical error in the proof came only at the very end, “This proves .” It had not sufficiently proven it, because it has proven only the cases that can be covered by building, which is not all cases.

**Problem 2: The Grow Algorithm**

**Theorem.** For any connected, weighted graph , ALG-GROW produces an MST of .

**a)** Prove the following lemma.

**Lemma 2.** Let be a tree and let be an edge not in . Then, contains a cycle.

**Proof.** By contradiction. Assume the lemma is false and does not contain a cycle. Since is a tree and therefore a connected acyclic graph, adding an edge to make means that it is still connected and still acyclic by the assumption, and therefore a tree. Furthermore, being a tree and by Lemma 1, has

edges. Since one edge is added to , must have edges. But then cannot be a tree because , thus there is a contradiction and Lemma 2 must be true.

**b)** Prove the following lemma.

**Lemma 3.** Let be a tree and let be an edge not in . Then there exists an edge in such that is a spanning tree of .

**Proof.** By contradiction. Assume there is not such an edge in . Then let be a step towards creating . By Lemma 2, must contain a cycle. Since did not contain a cycle, the cycle in must include . In order to be a cycle, the cycle must have at least two other edges than . We must show that removing one of these other edges creates a tree. A tree is connected and acyclic. Since this is the only cycle, and removing an edge can’t create any new cycles, removing will leave an acyclic graph. Now, let there be any two nodes that are currently connected. It could be that either is on the path that connects and or not. In the first case, the remaining edges of the cycle guarantee there is still a path between and . In the second case, removing ’ does not affect the path between and . Therefore, the graph is both acyclic and connected, making a tree, which is correspondingly a spanning tree of since it covers all of . Thus there is such an in and there is a contradiction, affirming Lemma 3.

**c)** Prove the following lemma.

**Lemma 4.** Let be a spanning tree of , let be an edge not in and let such that does not contain a cycle. Then there exists an edge in such that is a spanning tree of .

**Proof**. By contradiction. Assume there is not such an edge in . Then let be a step towards creating . By Lemma 2, must contain a cycle. Since did not contain a cycle, the cycle in must include . In order to be a cycle, the cycle must have at least two other edges than . Since does not have a cycle, one of those other edges must not be in . We must show that removing creates a tree. A tree is connected and acyclic. Since this is the only cycle, and removing an edge can’t create any new cycles, removing will leave an acyclic graph. Now, let there be any two nodes that are currently connected. It could be that either is on the path that connects and or not. In the first case, the remaining edges of the cycle guarantee there is still a path between and . In the second case, removing ’ does not affect the path between and . Therefore, the graph is both acyclic and connected, making a tree, which is correspondingly a spanning tree of since it covers all of . Thus there is such an in and there is a contradiction, affirming Lemma 3.

**d)** Prove the following lemma.

**Lemma 5.** Define to be the set consisting of the first edges selected by ALG-GROW from a connected graph . Let be the predicate that if then for some MST of . Then .

**Proof**. By induction on . Let be lemma 5. We will show such that if then for some MST of .

**Base**. . and the empty set is a subset of .

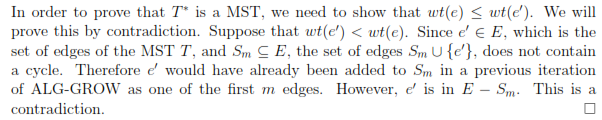
**Inductive step.** Assume such that the inductive hypothesis applies. Then, using ALG-GROW, the next smallest edge is selected that does not introduce a cycle, giving . If then the number of edges is greater than the number of nodes and could not be the subset of any tree. Correspondingly, ’s predicate is not met and holds. Then, if , there are two subcases.

1. The added edge is in for the MST . In this case, and holds.

2. The added edge is not in for the MST . Then we will show that it is in a different MST . Since is added with ALG-GROW, does not contain a cycle. Therefore, by Lemma 5, there exists an edge in such that is a spanning tree of . Thus for MST and holds.

In all cases therefore by the principle of induction Lemma 5 is true.

The part that I missed:



**e)** Prove the theorem.

For any connected, weighted graph , ALG-GROW produces an MST of .

**Proof.** Eventually ALG-GROW will have selected a set of precisely edges of . At this point, by Lemma 5, for some MST of . For the purposes of contradiction, assume is a proper subset of . Since describes the edges of a tree, . But also equals , so cannot be a proper subset of , therefore there is a contradiction and must be equal to . Thus ALG-GROW has produced a set of edges defining an MST of , and we are done. ∎